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ON HILL AND MOTT, TRANS. AMER. MATH. SOC. 175 (1973), 283–297. MR 47 #102

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The authors exhibit an abelian group not embeddable in a product of copies of Z (the integers) but totally orderable so that every covering pair of convex subgroups has archimedean quotient isomorphic to Z (= g.d.), thus furnishing a counterexample to Théorème 4 on p. 53 of [21]; then go on to show that every countable g.d. group is algebraically free (§4); and purport to show it also embeddable in a lexicographic "sum" (supposedly the subgroup of the product generated by the factors) of copies of Z (Theorem 5.1). This latter result is however itself in error, as follows from the internal characterization of "weak Hahn sums" [3] as those in which every element admits only convex decompositions of bounded length. Where any such, indeed even a g.d. uncountable free, group is order embeddable, is into the full lexicographic product of copies of Z; and in the more precise sense of reversing any splitting embeddings into it of its archimedean quotients. This follows from the original proof in [20], [21] whose only defect was to fail to verify preservation of addition; as a consequence of Corollary 4.3 moreover, countable g.d. groups will thereby necessarily be sent on images which are "almost direct" in the sense of [3, p. 434].

In more detail: in the direct sum, ordered lexicographically, of archimedean ordered groups over a totally ordered index set, an element with n nonzero components can have at most n nonzero components in any finite direct sum decomposition for which the sum of the first m subgroups is (for every m less than the number of summands) convex: indeed, the original sum refines this decomposition (no convex subgroup can split an archimedean one, hence the convex partial direct sums consist of unions of successively larger initial segments of the totally ordered under inclusion set of convex subgroups). Thus a countable g.d. group not embeddable in such a sum may be found in the lexicographic product of copies of Z indexed by the natural numbers N, by

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taking the subgroup generated by the individual factors and one other element: say the subgroup of cofinitely constant functions from N to Z ordered by first differences. In fact as a subgroup of the full lexicographic product this is "almost direct" [3]: i.e. it admits a direct sum decomposition at every index into three subgroups of which the proper convex partial direct sums are just the covering pair at that index: hence the possible number of nonzero components for any infinitely nonzero function will be unbounded. (More generally, the finitely nonzero elements contained in any almost direct subgroup of a lexicographic product of archimedean groups may be characterized as those having only a bounded number of possible nonzero components in such partially convex finite direct sum decompositions.)

To facilitate comparisons, the references are listed with the same numbers as in Hill and Mott.

References

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